

A BANACH SPACE WITH A COUNTABLE INFINITE NUMBER OF COMPLEX STRUCTURES

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ABSTRACT. We give examples of real Banach spaces with exactly infinite countably many complex structures and with ω_1 many complex structures.

1. INTRODUCTION

A real Banach space X is said to admit a complex structure when there exists a linear operator I on X such that $I^2 = -Id$. This turns X into a \mathbb{C} -linear space by declaring a new law for the scalar multiplication:

$$(\lambda + i\mu).x = \lambda x + \mu I(x) \quad (\lambda, \mu \in \mathbb{R}).$$

Equipped with the equivalent norm

$$\|x\| = \sup_{0 \leq \theta \leq 2\pi} \|\cos \theta x + \sin \theta Ix\|,$$

we obtain a complex Banach space which will be denoted by X^I . The space X^I is the complex structure of X associated to the operator I , which is often referred itself as a complex structure for X .

When the space X is already a complex Banach space, the operator $Ix = ix$ is a complex structure on $X_{\mathbb{R}}$ (i.e., X seen as a real space) which generates X . Recall that for a complex Banach space X its complex conjugate \overline{X} is defined to be the space X equipped with the new scalar multiplication $\lambda.x = \overline{\lambda}x$.

Two complex structures I and J on a real Banach space X are equivalent if there exists a real automorphism T on X such that $TI = JT$. This is equivalent to saying that the spaces X^I and X^J are \mathbb{C} -linearly isomorphic. To see this, simply observe that the relation $TI = JT$ actually means that the operator T is \mathbb{C} -linear as defined from X^I to X^J .

We note that a complex structure I on a real Banach space X is an automorphism whose inverse is $-I$, which is itself another complex structure on X . In fact, the complex space X^{-I} is the complex conjugate space of X^I . Clearly the spaces X^I and X^{-I} are always \mathbb{R} -linearly isometric. On the other hand, J. Bourgain [3] and N. Kalton [12] constructed

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examples of complex Banach spaces not isomorphic to their corresponding complex conjugates, hence these spaces admit at least two different complex structures. Bourgain example is an ℓ_2 sum of finite dimensional spaces whose distance to their conjugates tends to infinity. Kalton example is a twisted sum of two Hilbert spaces i.e., X has a closed subspace E such that E and X/E are Hilbertian, while X itself is not isomorphic to a Hilbert space.

Complex structures do not always exist on Banach spaces. The first example in the literature was the James space, proved by J. Dieudonné [4]. Other examples of spaces without complex structures are the uniformly convex space constructed by S. Szarek [15] and the hereditary indecomposable space of W. T. Gowers and B. Maurey [8]. Gowers [9, 10] also constructed a space with unconditional basis but without complex structures. In general these spaces have few operators. For example, every operator on the Gowers-Maurey space is a strictly singular perturbation of a multiple of the identity and this forbids complex structures: suppose that T is an operator on this space such that $T^2 = -Id$ and write $T = \lambda Id + S$ with S a strictly singular operator. It follows that $(\lambda^2 + 1)Id$ is strictly singular and of course this is impossible.

More examples of Banach spaces without complex structures were constructed by P. Koszmider, M. Martín and J. Merí [13, 14]. In fact, they introduced the notion of *extremely non-complex Banach space*: A real Banach space X is extremely non-complex if every bounded linear operator $T : X \rightarrow X$ satisfies the norm equality $\|Id + T^2\| = 1 + \|T\|^2$. Among their examples of extremely non complex spaces are $C(K)$ spaces with few operators (e.g. when every bounded linear operator T on $C(K)$ is of the form $T = gId + S$ where $g \in C(K)$ and S is a weakly compact operator on $C(K)$), a $C(K)$ space containing a complemented isomorphic copy of ℓ_∞ (thus having a richer space of operators than the first one mentioned) and an extremely non complex space not isomorphic to any $C(K)$ space.

Going back to the problem of uniqueness of complex structures, Kalton proved that spaces whose complexification is a primary space have at most one complex structure [6]. In particular, the classical spaces c_0 , ℓ_p ($1 \leq p \leq \infty$), $L_p[0, 1]$ ($1 \leq p \leq \infty$), and $C[0, 1]$ have a unique complex structure.

We have mentioned before examples of Banach spaces with at least two different complex structures. In fact, V. Ferenczi [5] constructed a space $X(\mathbb{C})$ such that the complex structure $X(\mathbb{C})^J$ associated to some operator J and its conjugate are the only complex structures on $X(\mathbb{C})$ up to isomorphism. Furthermore, every \mathbb{R} -linear operator T on $X(\mathbb{C})$ is of the form $T = \lambda Id + \mu J + S$, where λ, μ are reals and S is strictly singular. Ferenczi also proved that the space $X(\mathbb{C})^n$ has exactly $n + 1$ complex structures for every positive integer n . Going to the extreme, R. Anisca [1] gave examples of subspaces of L_p ($1 \leq p < 2$) which admit continuum many non-isomorphic complex structures.

The question remains about finding examples of Banach spaces with exactly infinite countably many different complex structures. A first natural approach to solve this problem is to construct an infinite sum of copies of $X(\mathbb{C})$, and in order to control the number of complex structures to take a regular sum, for instance, $\ell_1(X(\mathbb{C}))$. It follows that every \mathbb{R} -linear bounded operator T on $\ell_1(X(\mathbb{C}))$ is of the form $T = \lambda(T) + S$, where $\lambda(T)$ is the scalar part of T , i.e., an infinite matrix of operators on $X(\mathbb{C})$ of the form $\lambda_{i,j}Id + \mu_{i,j}J$, and S is an infinite matrix of strictly singular operators on $X(\mathbb{C})$. It is easy to prove that if T is a complex structure then $\lambda(T)$ is also a complex structure. Recall from [5] that two complex structures whose difference is strictly singular must be equivalent. Unfortunately, the operator S in the representation of T is not necessarily strictly singular, and this makes very difficult to understand the complex structures on $\ell_1(X(\mathbb{C}))$.

It is necessary to consider a more “rigid” sum of copies of spaces like $X(\mathbb{C})$. We found this interesting property in the space \mathfrak{X}_{ω_1} constructed by S. Argyros, J. Lopez-Abad and S. Todorcevic [2]. Based on that construction we present a separable reflexive Banach space $\mathfrak{X}_{\omega_2}(\mathbb{C})$ with exactly infinite countably many different complex structures which admits an infinite dimensional Schauder decomposition $\mathfrak{X}_{\omega_2}(\mathbb{C}) = \bigoplus_k \mathfrak{X}_k$ for which every \mathbb{R} -linear operator T on $\mathfrak{X}_{\omega_2}(\mathbb{C})$ can be written as $T = D_T + S$, where S is strictly singular, $D_T|_{\mathfrak{X}_k} = \lambda_k Id_{\mathfrak{X}_k}$ ($\lambda_k \in \mathbb{C}$) and $(\lambda_k)_k$ is a convergent sequence.

This construction also shows the existence of continuum many examples of Banach spaces with the property of having exactly ω complex structures and the existence of a Banach space with exactly ω_1 complex structures.

2. CONSTRUCTION OF THE SPACE $\mathfrak{X}_{\omega_1}(\mathbb{C})$

We construct a complex Banach space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ with a bimonotone transfinite Schauder basis $(e_\alpha)_{\alpha < \omega_1}$, such that every complex structure I on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is of the form $I = D + S$, where D is a suitable diagonal operator and S is strictly singular.

By a bimonotone transfinite Schauder basis we mean that $\mathfrak{X}_{\omega_1}(\mathbb{C}) = \overline{\text{span}}(e_\alpha)_{\alpha < \omega_1}$ and such that for every interval I of ω_1 the naturally defined map on the linear span of $(e_\alpha)_{\alpha < \omega_1}$

$$\sum_{\alpha < \omega_1} \lambda_\alpha e_\alpha \mapsto \sum_{\alpha \in I} \lambda_\alpha e_\alpha$$

extends to a bounded projection $P_I : \mathfrak{X}_{\omega_1}(\mathbb{C}) \rightarrow \mathfrak{X}_I = \overline{\text{span}}_{\mathbb{C}}(e_\alpha)_{\alpha \in I}$ with norm equal to 1.

Basically $\mathfrak{X}_{\omega_1}(\mathbb{C})$ corresponds to the complex version of the space \mathfrak{X}_{ω_1} constructed in [2] modifying the construction in a way that its \mathbb{R} -linear operators have similar structural properties to the operators in the original space \mathfrak{X}_{ω_1} (i.e. the operators are strictly singular perturbation of a complex diagonal operator).

First we introduce the notation that will be used through all this paper.

2.1. Basic notation. Recall that ω and ω_1 denotes the least infinite cardinal number and the least uncountable cardinal number, respectively. Given ordinals γ, ξ we write $\gamma + \xi, \gamma \cdot \xi, \gamma^\xi$ for the usual arithmetic operations (see [11]). For an ordinal γ we denote by

$\Lambda(\gamma)$ the set of limit ordinals $< \gamma$. Denote by $c_{00}(\omega_1, \mathbb{C})$ the vector space of all functions $x : \omega_1 \rightarrow \mathbb{C}$ such that the set $\text{supp } x = \{\alpha < \omega_1 : x(\alpha) \neq 0\}$ is finite and by $(e_\alpha)_{\alpha < \omega_1}$ its canonical Hamel basis. For a vector $x \in c_{00}(\omega_1, \mathbb{C})$ $\text{ran } x$ will denote the minimal interval containing $\text{supp } x$. Given two subsets E_1, E_2 of ω_1 we say that $E_1 < E_2$ if $\max E_1 < \min E_2$. Then for $x, y \in c_{00}(\omega_1, \mathbb{C})$ $x < y$ means that $\text{supp } x < \text{supp } y$. For a vector $x \in c_{00}(\omega_1, \mathbb{C})$ and a subset E of ω_1 we denote by Ex (or $P_E x$) the restriction of x on E or simply the function $x\chi_E$. Finally in some cases we shall denote elements of $c_{00}(\omega_1, \mathbb{C})$ as $f, g, h \dots$ and its canonical Hamel basis as $(e_\alpha^*)_{\alpha < \omega_1}$ meaning that we refer to these elements as being functionals in the norming set.

2.2. Definition of the norming set. The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ shall be defined as the completion of $c_{00}(\omega_1, \mathbb{C})$ equipped with a norm given by a norming set $\mathcal{K}_{\omega_1}(\mathbb{C}) \subseteq c_{00}(\omega_1, \mathbb{C})$. This means that the norm for every $x \in c_{00}(\omega_1, \mathbb{C})$ is defined as $\sup\{|\phi(x)| = |\sum_{\alpha < \omega_1} \phi(\alpha)x(\alpha)| : \phi \in \mathcal{K}_{\omega_1}(\mathbb{C})\}$. The norm of this space can also be defined inductively.

We start by fixing two fast increasing sequences (m_j) and (n_j) that are going to be used in the rest of this work. The sequences are defined recursively as follows:

1. $m_1 = 2$ e $m_{j+1} = m_j^4$;
2. $n_1 = 4$ e $n_{j+1} = (4n_j)^{s_j}$, where $s_j = \log_2 m_{j+1}^3$.

Let $\mathcal{K}_{\omega_1}(\mathbb{C})$ be the minimal subset of $c_{00}(\omega_1, \mathbb{C})$ such that

1. It contains every e_α^* , $\alpha < \omega_1$. It satisfies that for every $\phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and for every complex number $\theta = \lambda + i\mu$ with λ and μ rationals and $|\theta| \leq 1$, $\theta\phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$. It is closed under restriction to intervals of ω_1 .
2. For every $\{\phi_i, : i = 1, \dots, n_{2j}\} \subseteq \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $\phi_1 < \dots < \phi_{n_{2j}}$, the combination

$$\phi = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C}).$$

In this case we say that ϕ is the result of an (m_{2j}^{-1}, n_{2j}) -operation.

3. For every special sequence $(\phi_1, \dots, \phi_{n_{2j+1}})$ (see the Definition 13), the combination

$$\phi = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}} \phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C}).$$

In this case we say that ϕ is a special functional and that ϕ is the result of an $(m_{2j+1}^{-1}, n_{2j+1})$ -operation.

4. It is rationally convex.

Define a norm on $c_{00}(\omega_1, \mathbb{C})$ by setting

$$\|x\| = \sup \left\{ \left| \sum_{\alpha < \omega_1} \phi(\alpha)x(\alpha) \right| : \phi \in \mathcal{K}_{\omega_1}(\mathbb{C}) \right\}.$$

The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is defined as the completion of $(c_{00}(\omega_1, \mathbb{C}), \|\cdot\|)$.

This definition of the norming set $\mathcal{K}_{\omega_1}(\mathbb{C})$ is similar to the one in [2]. We add the property of being closed under products with rational complex numbers of the unit ball. This, together with property 2 above, guarantees the existence of some type of sequences (like ℓ_1^n -averages and *R.I.S* see Appendix) in the same way they are constructed for \mathfrak{X}_{ω_1} . It follows that the norm is also defined by

$$\|x\| = \sup \left\{ \phi(x) = \sum_{\alpha < \omega_1} \phi(\alpha)x(\alpha) : \phi \in \mathcal{K}_{\omega_1}(\mathbb{C}), \phi(x) \in \mathbb{R} \right\}.$$

We also have the following implicit formula for the norm:

$$\|x\| = \max \left\{ \|x\|_{\infty}, \sup_j \sup_{m_{2j}} \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \|E_i x\|, E_1 < E_2 < \dots < E_{n_{2j}} \right\} \vee \\ \sup \left\{ \frac{1}{m_{2j+1}} \left| \sum_{i=1}^{n_{2j+1}} \phi_i(Ex) \right| : (\phi_i)_{i=1}^{n_{2j+1}} \text{ is } n_{2j+1}\text{-special, } E \text{ interval} \right\}.$$

It follows from the definition of the norming set that the canonical Hamel basis $(e_{\alpha})_{\alpha < \omega_1}$ is a transfinite bimonotone Schauder basis of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. In fact, by Property 1 for every interval I of ω_1 the projection P_I has norm 1:

$$\|P_I x\| = \sup_{f \in \mathcal{K}_{\omega_1}(\mathbb{C})} |f P_I x| = \sup_{f \in \mathcal{K}_{\omega_1}(\mathbb{C})} |P_I f x| \leq \|x\|$$

Moreover, we have that the basis $(e_{\alpha})_{\alpha < \omega_1}$ is boundedly complete and shrinking, the proof is the obvious modification to the one for \mathfrak{X}_{ω_1} (see [2, Proposition 4.13]). In consequence $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is reflexive.

Proposition 1. $\overline{\mathcal{K}_{\omega_1}(\mathbb{C})}^{\omega^*} = B_{\mathfrak{X}_{\omega_1}^*(\mathbb{C})}.$

Proof. Recall that the set $\mathcal{K}_{\omega_1}(\mathbb{C})$ is by definition rational convex. We notice that $\overline{\mathcal{K}_{\omega_1}(\mathbb{C})}^{\omega^*}$ is actually a convex set. Indeed let $f, g \in \overline{\mathcal{K}_{\omega_1}(\mathbb{C})}^{\omega^*}$ and $t \in (0, 1)$. Suppose that $f_n \xrightarrow{\omega^*} f$, $g_n \xrightarrow{\omega^*} g$ and $t_n \rightarrow t$, where $f_n, g_n \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and $t_n \in \mathbb{Q} \cap (0, 1)$ for every $n \in \mathbb{N}$. Then $tf + (1-t)g \in \overline{\mathcal{K}_{\omega_1}(\mathbb{C})}^{\omega^*}$ because

$$t_n f_n + (1-t_n)g_n \xrightarrow{\omega^*} tf + (1-t)g.$$

In the same manner we can prove that $\mathfrak{X}_{\omega_1}^*(\mathbb{C})$ is balanced i.e., $\lambda \mathfrak{X}_{\omega_1}^*(\mathbb{C}) \subseteq \mathfrak{X}_{\omega_1}^*(\mathbb{C})$ for every $|\lambda| \leq 1$. To prove the Proposition suppose that there exists $f \in B_{\mathfrak{X}_{\omega_1}^*(\mathbb{C})} \setminus \overline{\mathcal{K}_{\omega_1}(\mathbb{C})}^{\omega^*}$. It follows by a standard separation argument that there exists $x \in \mathfrak{X}_{\omega_1}(\mathbb{C})$ such that

$$|f(x)| > \sup\{|g(x)| : g \in \mathcal{K}_{\omega_1}(\mathbb{C})\}$$

which is absurd. □

3. COMPLEX STRUCTURES ON $\mathfrak{X}_{\omega_1}(\mathbb{C})$

Let $I \subseteq \omega_1$ be an interval of ordinals, we denote by $\mathfrak{X}_I(\mathbb{C})$ the closed subspace of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ generated by $\{e_\alpha\}_{\alpha \in I}$. For every ordinal $\gamma < \omega_1$ we write $\mathfrak{X}_\gamma(\mathbb{C}) = \mathfrak{X}_{[0, \gamma)}(\mathbb{C})$. Notice that $\mathfrak{X}_I(\mathbb{C})$ is a 1-complemented subspace of $\mathfrak{X}_{\omega_1}(\mathbb{C})$: the restriction to coordinates in I is a projection of norm 1 onto $\mathfrak{X}_I(\mathbb{C})$. We denote this projection by P_I and by $P^I = (Id - P_I)$ the corresponding projection onto the complement space $(Id - P_I)\mathfrak{X}_{\omega_1}(\mathbb{C})$, which we denote by $\mathfrak{X}^I(\mathbb{C})$.

A transfinite sequence $(y_\alpha)_{\alpha < \gamma}$ is called a block sequence when $y_\alpha < y_\beta$ for all $\alpha < \beta < \gamma$. Given a block sequence $(y_\alpha)_{\alpha < \gamma}$ a *block subsequence* of $(y_\alpha)_{\alpha < \gamma}$ is a block sequence $(x_\beta)_{\beta < \xi}$ in the span of $(y_\alpha)_{\alpha < \gamma}$. A *real block subsequence* of $(y_\alpha)_{\alpha < \gamma}$ is a block subsequence in the real span of $(y_\alpha)_{\alpha < \gamma}$. A sequence $(x_n)_{n \in \mathbb{N}}$ is a block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ when it is a block subsequence of $(e_\alpha)_{\alpha < \omega_1}$.

Theorem 2. *Let $T : \mathfrak{X}_{\omega_1}(\mathbb{C}) \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$, that is, T is a bounded \mathbb{R} -linear operator such that $T^2 = -Id$. Then there exists a bounded diagonal operator $D_T : \mathfrak{X}_{\omega_1}(\mathbb{C}) \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$, which is another complex structure, such that $T - D_T$ is strictly singular. Moreover $D_T = \sum_{j=1}^k \epsilon_j i P_{I_j}$ for some signs $(\epsilon_j)_{j=1}^k$ and ordinal intervals $I_1 < I_2 < \dots < I_k$ whose extremes are limit ordinals and such that $\omega_1 = \bigcup_{j=1}^k I_j$.*

The strategy for the proof of Theorem 2 is the same than the one in [2, Theorem 5.32] for the real case. However here we want to understand bounded \mathbb{R} -linear operators in a complex space. This forces us to justify that the ideas from [2] still work in our context. The result is obtained using the following theorems that we explain with more details in the Appendix.

Step I. There exists a family \mathfrak{F} of semi normalized block subsequences of $(e_\alpha)_{\alpha < \omega_1}$, called *R.I.S (Rapidly Increasing Sequences)*, such that every normalized block sequence $(x_n)_{n \in \mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has a real block subsequence in \mathfrak{F} .

Recall that a Banach space X is hereditarily indecomposable (or H.I) if no (closed) subspace of X can be written as the direct sum of infinite-dimensional subspaces. Equivalently, for any two subspaces Y, Z of X and $\epsilon > 0$, there exist $y \in Y, z \in Z$ such that $\|y\| = \|z\| = 1$ and $\|y - z\| < \epsilon$.

Step II. For every normalized block sequence $(x_n)_{n \in \mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, the subspace $\overline{\text{span}}_{\mathbb{R}}(x_n)_{n \in \mathbb{N}}$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is a real H.I space.

Step III. Let $(x_n)_{n \in \mathbb{N}}$ be a *R.I.S* and $T : \overline{\text{span}}_{\mathbb{C}}(x_n)_{n \in \mathbb{N}} \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded \mathbb{R} -linear operator. Then $\lim_{n \rightarrow \infty} d(Tx_n, \mathbb{C}x_n) = 0$.

The proof of Step I, II and III are given in the Appendix.

Step IV. Let $(x_n)_{n \in \mathbb{N}}$ be a *R.I.S* and $T : \overline{\text{span}}_{\mathbb{C}}(x_n)_{n \in \mathbb{N}} \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded \mathbb{R} -linear

operator. Then the sequence $\lambda_T : \mathbb{N} \rightarrow \mathbb{C}$ defined by $d(Tx_n, \mathbb{C}x_n) = \|Tx_n - \lambda_T(n)x_n\|$ is convergent.

Proof of Step IV. First we note that the sequence $(\lambda_T(n))_n$ is bounded. Then consider $(\alpha_n)_n$ and $(\beta_n)_n$ two strictly increasing sequences of positive integers and suppose that $\lambda_T(\alpha_n) \rightarrow \lambda_1$ and $\lambda_T(\beta_n) \rightarrow \lambda_2$, when $n \rightarrow \infty$. Going to a subsequence we can assume that $x_{\alpha_n} < x_{\beta_n} < x_{\alpha_{n+1}}$ for every $n \in \mathbb{N}$.

Fix $\epsilon > 0$. Using the result of the Step III, we have that $\lim_{n \rightarrow \infty} \|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\| = 0$. By passing to a subsequence if necessary, assume

$$\|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\| \leq \frac{\epsilon}{2n6}.$$

for every $n \in \mathbb{N}$. Hence, for every $w = \sum_n a_n x_{\alpha_n} \in \text{span}_{\mathbb{R}}(x_{\alpha_n})_n$ with $\|w\| \leq 1$ we have

$$\begin{aligned} \|Tw - \lambda_1 w\| &\leq \sum_n |a_n| \|Tx_{\alpha_n} - \lambda_1 x_{\alpha_n}\| \\ &\leq \epsilon/3. \end{aligned}$$

because $(e_\alpha)_{\alpha < \omega_1}$ is a bimonotone transfinite basis. In the same way, we can assume that for every $w \in \text{span}_{\mathbb{R}}(x_{\beta_m})_m$ with $\|w\| \leq 1$, $\|Tw - \lambda_2 w\| \leq \epsilon/3$. By Step II we have that $\overline{\text{span}}_{\mathbb{R}}(x_{\alpha_n})_n \cup (x_{\beta_n})_n$ is real-H.I. Then there exist unit vectors $w_1 \in \text{span}_{\mathbb{R}}(x_{\alpha_n})_n$ and $w_2 \in \text{span}_{\mathbb{R}}(x_{\beta_m})_m$, such that $\|w_1 - w_2\| \leq \frac{\epsilon}{3}\|T\|$. Therefore,

$$\|\lambda_1 w_1 - \lambda_2 w_2\| \leq \|Tw_1 - \lambda_1 w_1\| + \|Tw_1 - Tw_2\| + \|Tw_2 - \lambda_2 w_2\| \leq \epsilon.$$

By other side

$$\|\lambda_1 w_1 - \lambda_2 w_2\| \geq \|(\lambda_1 - \lambda_2)w_1\| - \|\lambda_2(w_1 - w_2)\| = |\lambda_1 - \lambda_2| - |\lambda_2|\epsilon.$$

In consequence, $|\lambda_1 - \lambda_2| \leq (1 + |\lambda_2|)\epsilon$. Since ϵ was arbitrary, it follows that $\lambda_1 = \lambda_2$. \square

Let $T : \mathfrak{X}_{\omega_1}(\mathbb{C}) \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded \mathbb{R} -linear operator. There is a canonical way to associate a bounded diagonal operator D_T (with respect to the basis $(e_\gamma)_{\gamma < \omega_1}$) such that $T - D_T$ is strictly singular: Fix $\alpha \in \Lambda(\omega_1)$ a limit ordinal, and $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ two *R.I.S* such that $\sup_n \max \text{supp } x_n = \sup_n \max \text{supp } y_n = \alpha + \omega$. By a property of \mathfrak{F} we can mix the sequences $(x_n)_n$, $(y_n)_n$ in order to form a new *R.I.S* $(z_n)_{n \in \mathbb{N}}$ such that $z_{2k} \in \{x_n\}_{n \in \mathbb{N}}$ and $z_{2k-1} \in \{y_n\}_{n \in \mathbb{N}}$ for all $k \in \mathbb{N}$ (See Remark 16). Then it follows from Step IV that the sequences defined by the formulas $d(Tx_n, \mathbb{C}x_n) = \|Tx_n - \lambda_T(n)x_n\|$ and $d(Ty_n, \mathbb{C}y_n) = \|Ty_n - \mu(n)y_n\|$ are convergent, and by the mixing argument, they must have the same limit. Hence for each $\alpha \in \Lambda(\omega_1)$ there exists a unique complex number $\xi_T(\alpha)$ such that

$$\lim_{n \rightarrow \infty} \|Tw_n - \xi_T(\alpha)w_n\| = 0$$

for every $(w_n)_{n \in \mathbb{N}}$ *R.I.S* in \mathfrak{X}_{I_α} , where we write I_α to denote the ordinal interval $[\alpha, \alpha + \omega)$. We proceed to define a diagonal linear operator D_T on the (linear) decomposition of

$\text{span}(e_\alpha)_{\alpha < \omega_1}$

$$\text{span}(e_\alpha)_{\alpha < \omega_1} = \bigoplus_{\alpha \in \Lambda(\omega_1)} \text{span}(x_\beta)_{\beta \in I_\alpha}$$

by setting $D_T(e_\beta) = \xi_T(\alpha)e_\beta$ when $\beta \in I_\alpha$.

Observe in addition that this sequence $(\xi_T(\alpha))_{\alpha \in \Lambda(\omega_1)}$ is convergent. That is, for every strictly increasing sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $\Lambda(\omega_1)$, the corresponding subsequence $(\xi_T(\alpha_n))_{n \in \mathbb{N}}$ is convergent. In fact, for every $n \in \mathbb{N}$, fix $(y_n^k)_{k \in \mathbb{N}}$ a *R.I.S* in $\mathfrak{X}_{I_{\alpha_n}}$. Then we can take $(y_n^k)_{n \in \mathbb{N}}$ a *R.I.S* such that $\|Ty_n^k - \xi_T(\alpha_n + \omega)y_n^k\| < 1/n$. It follows by Step IV there exists $\lambda \in \mathbb{C}$ such that $\lim_n \|Ty_n^k - \lambda y_n^k\| = 0$. This implies that $\lim_n \xi_T(\alpha_n + \omega) = \lambda$.

In general this operator D_T defines a bounded operator on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. The proof is the same that in [2, Proposition 5.31] and uses that certain James like space of a mixed Tsirelson space is finitely interval representable in every normalized transfinite block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. For the case of complex structures we have a simpler proof (see Proposition 6).

Proposition 3. *Let A be a subset of ordinals contained in ω_1 and $X = \overline{\text{span}}_{\mathbb{C}}(e_\alpha)_{\alpha \in A}$. Let $T : X \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded \mathbb{R} -linear operator. Then T is strictly singular if and only if for every $(y_n)_{n \in \mathbb{N}}$ *R.I.S* on X , $\lim_n Ty_n = 0$.*

Proof. The proposition is trivial when the set A is finite, then we assume that A is infinite. Suppose that T is strictly singular. Let $(y_n)_{n \in \mathbb{N}}$ be a *R.I.S* on X such that $\lim_n Ty_n \neq 0$, then by Step IV there is $\lambda \neq 0$ with $\lim_n \|Ty_n - \lambda y_n\| = 0$. Take $0 < \epsilon < |\lambda|$. By passing to a subsequence if necessary, we assume that $\|(T - \lambda Id)|_{\overline{\text{span}}(y_n)_n}\| < \epsilon$. This implies that $T|_{\overline{\text{span}}(y_n)_n}$ is an isomorphism which is a contradiction.

Conversely, suppose that for every $(y_n)_n$ *R.I.S* on X , $\lim_n Ty_n = 0$. Assume that T is not strictly singular. Then there is a block sequence subspace $Y = \overline{\text{span}}(y_n)_{n \in \mathbb{N}}$ of X such that T restricted to Y is an isomorphism. By Step I we can assume that the sequence $(y_n)_n$ is already a *R.I.S* on X . Then $\inf_n \|Ty_n\| > 0$. And we obtain a contradiction. \square

Given $Y \subseteq \mathfrak{X}_{\omega_1}(\mathbb{C})$ we denote by ι_Y the canonical inclusion of Y into $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

Corollary 4. *Let $\alpha \in \Lambda(\omega_1)$ and $T : \mathfrak{X}_{I_\alpha}(\mathbb{C}) \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded \mathbb{R} -linear operator. Then there exists (unique) $\xi_T(\alpha) \in \mathbb{C}$ such that $T - \xi_T(\alpha)\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$ is strictly singular.*

Proof. Let $\xi_T(\alpha)$ be the (unique) complex number such that $\lim \|Ty_n - \xi_T(\alpha)y_n\| = 0$ for every $(y_n)_n$ *R.I.S* on $\mathfrak{X}_{I_\alpha}(\mathbb{C})$. Then by the previous Proposition $T - \xi_T(\alpha)\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$ is strictly singular. \square

Corollary 5. *Let $\alpha \in \Lambda(\omega_1)$ and $R : \mathfrak{X}_{I_\alpha}(\mathbb{C}) \rightarrow \mathfrak{X}^{I_\alpha}(\mathbb{C})$ be a bounded \mathbb{R} -linear operator. Then R is strictly singular.*

Proof. By the previous result, $\iota_{\mathfrak{X}^{I_\alpha}(\mathbb{C})}R = \lambda_\alpha \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S$ with S strictly singular. Then projecting by P^{I_α} we obtain $R = P^{I_\alpha} \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}R = P^{I_\alpha}S$ which is strictly singular. \square

Proposition 6. *Let T be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then the linear operator D_T is a bounded complex structure.*

Proof. Let T be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and D_T the corresponding diagonal operator defined above. Fix $\alpha \in \Lambda(\omega_1)$. We shall prove that $\xi_T(\alpha)^2 = -1$. In fact,

$$\begin{aligned} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} &= P_{I_\alpha} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + P^{I_\alpha} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} \\ &= P_{I_\alpha} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_1 \end{aligned}$$

where S_1 is strictly singular. This implies $P_{I_\alpha} T \circ \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} = \xi_T(\alpha) Id_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_2 : \mathfrak{X}_{I_\alpha}(\mathbb{C}) \rightarrow \mathfrak{X}_{I_\alpha}(\mathbb{C})$ with S_2 strictly singular. Now computing:

$$\begin{aligned} (P_{I_\alpha} T \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}) \circ (P_{I_\alpha} T \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}) &= P_{I_\alpha} T \circ P_{I_\alpha} T \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} \\ &= P_{I_\alpha} T \circ (Id - P^{I_\alpha}) T \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} \\ &= P_{I_\alpha} T^2 \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} - P_{I_\alpha} T \underline{P^{I_\alpha} T \iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}} \\ &= -Id_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S_3 \end{aligned}$$

where S_3 is strictly singular because the underlined operator is strictly singular. Hence we have that $(\xi_T(\alpha)^2 + 1) Id_{\mathfrak{X}_{I_\alpha}(\mathbb{C})}$ is strictly singular. Which allow us to conclude that $\xi_T(\alpha)^2 = -1$. The continuity of D_T is then guaranteed by the convergence of $(\xi_T(\alpha))_{\alpha \in \Lambda(\omega_1)}$. In deed, we have that there exist ordinal intervals $I_1 < I_2 < \dots < I_k$ with $\omega_1 = \bigcup_{j=1}^k I_j$ and such that $D_T = \sum_{j=1}^k \epsilon_j i P_{I_j}$ for some signs $(\epsilon_j)_{j=1}^k$. \square

Remark 7. *More generally, the proof of Proposition 6 actually shows that if T is a \mathbb{R} -linear bounded operator on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that $T^2 + Id = S$ for some S strictly singular, then D_T is bounded and $D_T^2 = -Id$.*

Now we can conclude the proof of Theorem 2.

Proof of Theorem 2. Let $T : \mathfrak{X}_{\omega_1}(\mathbb{C}) \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ be a bounded \mathbb{R} -linear operator which is a complex structure and D_T be the diagonal bounded operator associated to it. It only remains to prove that $T - D_T$ is strictly singular. And this follows directly from Proposition 3, because by definition $\lim_n (T - D_T) y_n = 0$ for every $(y_n)_n$ R.I.S on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. \square

We come back to the study of the complex structures on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Denote by \mathfrak{D} the family of complex structures D_T on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ as in Theorem 2, i. e., $D_T = \sum_{j=1}^k \epsilon_j i P_{I_j}$ where $(\epsilon_j)_{j=1}^k$ are signs and $I_1 < I_2 < \dots < I_k$ are ordinal intervals whose extremes are limit ordinals and such that $\omega_1 = \bigcup_{j=1}^k I_j$. Notice that \mathfrak{D} has cardinality ω_1 .

Recall that two spaces are said to be incomparable if neither of them embed into the other.

Corollary 8. *The space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has ω_1 many complex structures up to isomorphism. Moreover any two non-isomorphic complex structures are incomparable.*

Proof. Let J be a complex structure on $\mathfrak{X}_{\omega_1}(\mathbb{C})$. By Theorem 2 we have that J is equivalent to one of the complex structures of the family \mathfrak{D} .

To complete the proof it is enough to show that given two different elements of \mathfrak{D} they define non equivalent complex structures. Moreover, we prove that one structure does not embed into the other. Fix $J \neq K \in \mathfrak{D}$. Then there exists an ordinal interval $I_\alpha = [\alpha, \alpha + \omega)$ such that, without loss of generality, $J|_{\mathfrak{X}_{I_\alpha}} = iId|_{\mathfrak{X}_{I_\alpha}}$ and $K|_{\mathfrak{X}_{I_\alpha}} = -iId|_{\mathfrak{X}_{I_\alpha}}$. Suppose that there exists $T : \mathfrak{X}_{\omega_1}(\mathbb{C})^J \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})^K$ an isomorphic embedding. Then T is in particular a \mathbb{R} -linear operator such that $TJ = KT$. We write using Corollary 4, $T|_{\mathfrak{X}_{I_\alpha}} = \xi_T(\alpha)\iota_{\mathfrak{X}_{I_\alpha}(\mathbb{C})} + S$ with S strictly singular. Then $\xi_T(\alpha)J|_{\mathfrak{X}_{I_\alpha}} - \xi_T(\alpha)K|_{\mathfrak{X}_{I_\alpha}} = S_1$ where S_1 is strictly singular. In particular for each $x \in \mathfrak{X}_{I_\alpha}$, $S_1x = 2\xi_T(\alpha)ix$. It follows from the fact that \mathfrak{X}_{I_α} is infinite dimensional that $\xi_T(\alpha) = 0$. Hence $T|_{\mathfrak{X}_{I_\alpha}} = S$ but this is a contradiction because T is an isomorphic embedding. \square

The next corollary offers uncountably many examples of Banach spaces with exactly countably many complex structures.

Corollary 9. *The space $\mathfrak{X}_\gamma(\mathbb{C})$ has ω complex structures up to isomorphism for every limit ordinal $\omega^2 \leq \gamma < \omega_1$.*

Proof. Let J be a complex structure on $\mathfrak{X}_\gamma(\mathbb{C})$. We extend J to a complex structure defined in the whole space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ by setting $T = JP_I + iP^I$, where $I = [0, \gamma)$. It follows that $T = D_T + S$ for an strictly singular operator S and a diagonal operator D_T like in Theorem 2. Notice that $D_Tx = ix$ for every $x \in \mathfrak{X}^I$, otherwise there would be a limit ordinal α such that $S|_{\mathfrak{X}_{I_\alpha}} = 2iId|_{\mathfrak{X}_{I_\alpha}}$. Hence $JP_I = D_TP_I + S$. Which implies that J has the form $J = \sum_{j=1}^k \epsilon_j i P_{I_j} + S_1$ where S_1 is strictly singular on $\mathfrak{X}_{\omega_1}(\mathbb{C})$, $(\epsilon_j)_{j=1}^k$ are signs and $I_1 < I_2 < \dots < I_k$ are ordinal intervals whose extremes are limit ordinals and such that $\gamma = \cup_{j=1}^k I_j$. Now the rest of the proof is identical to the proof of the previous corollary. In particular, all the non-isomorphic complex structures on $\mathfrak{X}_\gamma(\mathbb{C})$ are incomparable. \square

We also have, using the same proof of the previous corollary, that for every increasing sequence of limit ordinals $A = (\alpha_n)_n$, the space $\mathfrak{X}_A = \bigoplus_n \mathfrak{X}_{I_{\alpha_n}}(\mathbb{C})$, where $I_{\alpha_n} = [\alpha_n, \alpha_n + \omega)$, has exactly infinite countably many different complex structures. Hence there exists a family, with the cardinality of the continuum, of Banach spaces such that every space in it has exactly ω complex structures.

4. QUESTION AND OBSERVATIONS

Is easy to check that subspaces of even codimension of a real Banach space with complex structure also admit complex structure. An interesting property of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is that any of its real hyperplanes (and thus every real subspace of odd codimension) do not admit complex structure.

Proposition 10. *The real hyperplanes of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ do not admit complex structure.*

Proof. By the results of Ferenczi and E. Galego [7, Proposition 13] it is sufficient to prove that the ideal of all \mathbb{R} -linear strictly singular operators on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ has the lifting property, that is, for any \mathbb{R} -linear isomorphism on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that $T^2 + Id$ is strictly singular, there exists a strictly singular operator S such that $(T - S)^2 = -Id$. The proof now follows easily from the Remark 7. \square

We now pass to present some open questions related to the results exposed in this paper. The first question is about a remark mentioned in the introduction and the Ferenczi's space $X(\mathbb{C})$ with exactly two complex structures.

Question 1. For every $1 \leq p < \infty$. How many complex structures has $\ell_p(X(\mathbb{C}))$?

Clearly the space $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is non separable. Hence a natural question is:

Question 2. Does there exist a separable Banach space with exactly ω_1 complex structures?

Question 3. Does there exist for every infinite cardinal κ a Banach space with κ -many non-equivalent complex structures?

One open problem in the theory of complex structure is to know if the existence of more regularity in the space guarantees that it admits unique complex structure.

Question 4. Does there exist a real Banach space with unconditional basis admitting more than one complex structure?

The question is still interesting in spaces with even more regularity than an unconditional basis. For example, when a real Banach space X has a symmetric basis. In this case, X admits at least one complex structure, because it is isomorphic to its square.

Question 5 Does every real Banach space with symmetric basis have unique complex structure?

Question 5 is strongly related with the well-known open problem: *Is every Banach space X , with a symmetric basis, primary?* In fact, a positive answer to this problem implies a positive solution for Question 5. We just have to note that the complexification of a space with symmetric basis has a symmetric basis, and recall Kalton's result: a Banach space such that its complexification is a primary space has unique complex structure.

5. APPENDIX

The purpose of this section is to give a proof for the results in the Step I, II and III. Several proofs are very similar to the corresponding ones in [2]. In order to make this paper as self contained as possible, we reproduce them in detail.

First we clarify the definition of the norming set by defining what being a special sequence means. All the definitions we present in this part are the corresponding translation of [2] for the complex case.

5.1. Coding and Special sequences. Recall that $[\omega_1]^2 = \{(\alpha, \beta) \in \omega_1^2 : \alpha < \beta\}$.

Definition 11. A function $\varrho : [\omega_1]^2 \rightarrow \omega$ such that

- (1) $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.
- (2) $\varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$ for all $\alpha < \beta < \gamma < \omega_1$.
- (3) The set $\{\alpha < \beta : \varrho(\alpha, \beta) \leq n\}$ is finite for all $\beta < \omega_1$ and $n \in \mathbb{N}$

is called a ϱ -function.

The existence of ϱ -functions is due to Todorćević [16]. Let us fix a ϱ -function $\varrho : [\omega_1]^2 \rightarrow \omega$ and all the following work relies on that particular choice of ϱ .

Definition 12. Let F be a finite subset of ω_1 and $p \in \mathbb{N}$, we write

$$\rho_F = \rho_\varrho(F) = \max_{\alpha, \beta \in F} \varrho(\alpha, \beta).$$

$$\overline{F}^p = \{\alpha \leq \max F : \text{there is } \beta \in F \text{ such that } \alpha \leq \beta \text{ and } \varrho(\alpha, \beta) \leq p\}$$

σ_ϱ -coding and the special sequences

We denote by $\mathbb{Q}_s(\omega_1, \mathbb{C})$ the set of finite sequences $(\phi_1, w_1, p_1, \dots, \phi_d, w_d, p_d)$ such that

- (1) For all $i \leq d$, $\phi_i \in c_{00}(\omega_1, \mathbb{C})$ and for all $\alpha < \omega_1$ the real and the imaginary part of $\phi(\alpha)$ are rationals.
- (2) $(w_i)_{i=1}^d, (p_i)_{i=1}^d \in \mathbb{N}^d$ are strictly increasing sequences.
- (3) $p_i \geq \rho(\cup_{k=1}^i \text{supp } \phi_k)$ for every $i \leq d$.

Let $\mathbb{Q}_s(\mathbb{C})$ be the set of finite sequences $(\phi_1, w_1, p_1, \phi_2, w_2, p_2, \dots, \phi_d, w_d, p_d)$ satisfying properties (1), (2) above and for every $i \leq d$, $\phi_i \in c_{00}(\omega, \mathbb{C})$. Then $\mathbb{Q}_s(\mathbb{C})$ is a countable set while $\mathbb{Q}_s(\omega_1, \mathbb{C})$ has cardinality ω_1 . Fix a one to one function $\sigma : \mathbb{Q}_s(\mathbb{C}) \rightarrow \{2j : j \text{ is odd}\}$ such that

$$\sigma(\phi_1, w_1, p_1, \dots, \phi_d, w_d, p_d) > \max\{p_d^2, \frac{1}{\epsilon^2}, \max \text{supp } \phi_d\}$$

where $\epsilon = \min\{|\phi_k(e_\alpha)| : \alpha \in \text{supp } \phi_k, k = 1, \dots, d\}$. Given a finite subset F of ω_1 , we denote by $\pi_F : \{1, 2, \dots, \#F\} \rightarrow F$ the natural order preserving map, i.e. π_F is the increasing numeration of F .

Given $\Phi = (\phi_1, w_1, p_1, \dots, \phi_d, w_d, p_d) \in \mathbb{Q}_s(\mathbb{C})$, we set

$$G_\Phi = \overline{\cup_{i=1}^d \text{supp } \phi_i}^{p_d}.$$

Consider the family $\pi_{G_\Phi}(\Phi) = (\pi_G(\phi_1), w_1, p_1, \pi_G(\phi_2), w_2, p_2, \dots, \pi_G(\phi_d), w_d, p_d)$ where

$$\pi_G(\phi_k)(n) = \begin{cases} \phi_k(\pi_{G_\Phi}(n)), & \text{if } n \in G_\Phi \\ 0, & \text{otherwise.} \end{cases}$$

Finally $\sigma_p : \mathbb{Q}_s(\omega_1, \mathbb{C}) \rightarrow \{2j : j \text{ odd}\}$ is defined by $\sigma_p(\Phi) = \sigma(\pi_G(\Phi))$.

Definition 13. A sequence $\Phi = (\phi_1, \phi_2, \dots, \phi_{n_{2j+1}})$ of functionals of $\mathcal{K}_{\omega_1}(\mathbb{C})$ is called a $2j+1$ special sequence if

(SS 1.) $\text{supp } \phi_1 < \text{supp } \phi_2 < \dots < \text{supp } \phi_{n_{2j+1}}$. For each $k \leq n_{2j+1}$, ϕ_k is of type I, $w(\phi_k) = m_{2j_k}$ with j_1 even and $m_{2j_1} > n_{2j+1}^2$.

(SS 2.) There exists a strictly increasing sequence $(p_1^\Phi, p_2^\Phi, \dots, p_{n_{2j+1}-1}^\Phi)$ of naturals numbers such that for all $1 \leq i \leq n_{2j+1} - 1$ we have that $w(\phi_{i+1}) = m_{\sigma_\varrho(\Phi_i)}$ where

$$\Phi_i = (\phi_1, w(\phi_1), p_1^\Phi, \phi_2, w(\phi_2), p_2^\Phi, \dots, \phi_i, w(\phi_i), p_i^\Phi)$$

Special sequences in separable examples with one to one codings are in general simpler: they are of the form $(\phi_1, w(\phi_1), \dots, \phi_k, w(\phi_k))$. Their main feature is that if $(\phi_1, w(\phi_1), \dots, \phi_k, w(\phi_k))$ and $(\psi_1, w(\psi_1), \dots, \psi_l, w(\psi_l))$ are two of them, there exists $i_0 \leq \min\{k, l\}$ with the property that

- (1) $(\phi_i, w(\phi_i)) = (\psi_i, w(\psi_i))$ for all $i \leq i_0$
- (2) $\{w(\phi_i) : i_0 \leq i \leq k\} \cap \{w(\psi_i) : i_0 \leq i \leq l\} = \emptyset$

In non-separable spaces, one to one codings are obviously impossible, and (1), (2) are no longer true. Fortunately, there is a similar feature to (1), (2) called the tree-like interference of a pair of special sequences (See [2, Lemma 2.9]): Let $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$ and $\Psi = (\psi_1, \dots, \psi_{n_{2j+1}})$ be two $2j+1$ -special sequences, then there exist two numbers $0 \leq \kappa_{\Phi, \Psi} \leq \lambda_{\Phi, \Psi} \leq n_{2j+1}$ such that the following conditions hold:

TP. 1 For all $i \leq \lambda_{\Phi, \Psi}$, $w(\phi_i) = w(\psi_i)$ and $p_i^\Phi = p_i^\Psi$.

TP. 2 For all $i < \kappa_{\Phi, \Psi}$, $\phi_i = \psi_i$.

TP. 3 For all $\kappa_{\Phi, \Psi} < i < \lambda_{\Phi, \Psi}$

$$\text{supp } \phi_i \cap \overline{\text{supp } \psi_1 \cup \dots \cup \text{supp } \psi_{\lambda_{\Phi, \Psi}-1}}^{p_{\lambda_{\Phi, \Psi}}^{-1}} = \emptyset$$

$$\text{and } \text{supp } \psi_i \cap \overline{\text{supp } \phi_1 \cup \dots \cup \text{supp } \phi_{\lambda_{\Phi, \Psi}-1}}^{p_{\lambda_{\Phi, \Psi}}^{-1}} = \emptyset$$

TP. 4 $\{w(\phi_i) : \lambda_{\Phi, \Psi} < i \leq n_{2j+1}\} \cap \{w(\psi_i) : i \leq n_{2j+1}\} = \emptyset$ and $\{w(\psi_i) : \lambda_{\Phi, \Psi} < i \leq n_{2j+1}\} \cap \{w(\phi_i) : i \leq n_{2j+1}\} = \emptyset$.

5.2. Rapidly increasing sequences (R.I.S). For the proof of Step I we shall construct a family of block sequences on $\mathfrak{X}_{\omega_1}(\mathbb{C})$ commonly called *rapidly increasing sequences (R.I.S)*. These sequences are very useful because one has good estimates of upper bounds on $|f(x)|$ for $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and x averages of R.I.S.

For the construction of the family \mathfrak{F} the only difference from the general theory in [2] is that our interest now is to study bounded \mathbb{R} -linear operators on the complex space $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Hence, all the construction of R.I.S in a particular block sequence $(x_n)_{n \in \mathbb{N}}$ must be on its real linear span. We point out here that there are no problems with this, because all the combinations of the vectors $(x_n)_{n \in \mathbb{N}}$ to obtain R.I.S use rational scalars.

Definition 14 (R.I.S). We say that a block sequence $(x_k)_k$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ is a (C, ϵ) -R.I.S, $C, \epsilon > 0$, when there exists a strictly increasing sequence of natural numbers $(j_k)_k$ such that:

- (i) $\|x_k\| \leq C$;

- (ii) $|\text{supp } x_k| \leq m_{j_{k+1}} \epsilon;$
- (iii) For all the functionals ϕ of $\mathcal{K}_{\omega_1}(\mathbb{C})$ of type I, with $\omega(\phi) < m_{j_k}$, $|\phi(x_k)| \leq \frac{C}{\omega(\phi)}.$

The following remark is immediately consequence of this definition.

Remark 15. Let $\epsilon' < \epsilon$. Every (C, ϵ) -R.I.S has a subsequence which is a (C, ϵ') -R.I.S. And for every strictly increasing sequence of ordinals $(\alpha_n)_n$ and every $\epsilon > 0$, $(e_{\alpha_n})_n$ is a $(1, \epsilon)$ -R.I.S.

Remark 16. Let $(x_n)_n$ and $(y_n)_n$ be two (C, ϵ) - R.I.S such that $\sup_n \max \text{supp } x_n = \sup_n \max \text{supp } y_n$. Then there exists $(z_n)_n$ a (C, ϵ) - R.I.S. such that $z_{2n-1} \in \{x_k\}_{k \in \mathbb{N}}$ and $z_{2n} \in \{y_k\}_{k \in \mathbb{N}}$.

Proof. Suppose that $(t_k)_k$ and $(s_k)_k$ are increasing sequences of positive integers satisfying the definition of R.I.S for $(x_k)_k$ and $(y_k)_k$ respectively. We construct $(z_k)_k$ as follows. Let $z_1 = x_1$ and $j_1 = t_1$. Pick s_{k_1} such that $x_1 < y_{s_{k_1}}$ and $t_2 < s_{k_1}$. Then we define $j_2 = s_{k_1}$ and $z_2 = y_{s_{k_1}}$. Notice that

- (i) $\|z_1\| \leq C;$
- (ii) $|\text{supp } z_1| \leq m_{t_2} \epsilon \leq m_{s_{k_1}} \epsilon = m_{j_2} \epsilon;$
- (iii) For all the functionals ϕ of $\mathcal{K}_{\omega_1}(\mathbb{C})$ of type I, with $\omega(\phi) < m_{j_1}$, $|\phi(z_1)| \leq \frac{C}{\omega(\phi)}.$

Continuing with this process we obtain the desired sequence. □

Theorem 17. Let $(x_k)_k$ be a normalized block sequence of \mathfrak{X}_{ω_1} and $\epsilon > 0$. Then there exists a normalized block subsequence $(y_n)_n$ in $\text{span}_{\mathbb{R}}\{x_k\}$ which is a $(3, \epsilon) - \text{R.I.S.}$

For the proof of Theorem 17 we first construct a simpler type of sequence.

Definition 18. Let X be a Banach space, $C \geq 1$ and $k \in \mathbb{N}$. A normalized vector y is called a $C - \ell_1^k$ -average of X , when there exist a block sequence (x_1, \dots, x_k) such that

- (1) $y = (x_1 + \dots + x_k)/k;$
- (2) $\|x_i\| \leq C$, for all $i = 1, \dots, k$

In the next result we want to emphasize that this special type of sequence are really constructed on the real structure of the space $\mathfrak{X}_{\omega_1}(\mathbb{C})$.

Theorem 19. For every normalized block sequence (x_n) of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every integer k , there exist $z_1 < \dots < z_k$ in $\text{span}_{\mathbb{R}}(x_n)$, such that $(z_1 + \dots + z_k)/k$ is a $2 - \ell_1^k$ -average.

Proof. The proof is standard. Suppose that the result is false. Let j and n be natural numbers with

$$\begin{aligned} 2^n &> m_{2j} \\ n_{2j} &> k^n. \end{aligned}$$

Let $N = k^n$ and $x = \sum_{i=1}^N x_i$. For each $1 \leq i \leq n$ and every $1 \leq j \leq k^{n-i}$, we define,

$$x(i, j) = \sum_{t=(j-1)k^i+1}^{jk^i} x_t.$$

Hence, $x(0, j) = x_j$ and $x(n, 1) = x$.

It is proved by induction on i that $\|x(i, j)\| \leq 2^{-i}k^i$, for all i, j . In particular, $\|x\| = \|x(n, 1)\| \leq 2^{-n}k^n = 2^{-n}N$. Then by *Property 1.* of definition in the norming set

$$\|x\| \geq \frac{1}{m_{2j}} \sum_{t=1}^{n_{2j}} \|x_t\| = \frac{n_{2j}}{m_{2j}} > \frac{N}{m_{2j}}.$$

Hence,

$$\begin{aligned} 2^{-n}N &> \frac{N}{m_{2j}} \\ m_{2j} &> 2^n, \end{aligned}$$

which is a contradiction. \square

Finally, for the construction of *R.I.S* we observe these simple facts ([2, Remark 4.10])

- If y is a $C - \ell_1^{n_j}$ -average of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $\phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ has weight $\omega(\phi) < m_j$, then $|\phi(y)| \leq \frac{3C}{2\omega(\phi)}$;
- Let $(x_k)_k$ be a block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ such that there exists a strictly increasing sequence of positive integers $(j_k)_k$ and $\epsilon > 0$ satisfying:
 - a) Each x_k is a $2 - \ell_1^{n_{j_k}}$ -average;
 - b) $|supp x_k| < \epsilon m_{j_{k+1}}$.

Then $(x_k)_k$ is a $(3, \epsilon) - R.I.S$.

5.3. Basic Inequality. To prove Step II and III we need a crucial result called *the basic inequality* which is very important to find good estimations for the norm of certain combinations of *R.I.S* in $\mathfrak{X}_{\omega_1}(\mathbb{C})$. First we need to introduce the *mixed Tsirelson spaces*.

The mixed Tsirelson space $T[(m_j^{-1}, n_j)_j]$ is defined by considering the completion of $c_{00}(\omega, \mathbb{C})$ under the norm $\|\cdot\|_0$ given by the following implicit formula

$$\|x\|_0 = \max \left\{ \|x\|_\infty, \sup_j \sup \frac{1}{m_j} \sum_{i=1}^{n_j} \|E_i x\|_0 \right\},$$

The supremum inside the formula is taken over all the sequences $E_1 < \dots < E_{n_j}$ of subsets of ω . Notice that in this space the canonical Hamel basis $(e_n)_{n < \omega}$ of $c_{00}(\omega, \mathbb{C})$ is 1-subsymmetric and 1-unconditional basis.

We can give an alternative definition for the norm of $T[(m_j^{-1}, n_j)_j]$ by defining the following norming set. Let $W[(m_j^{-1}, n_j)] \subseteq c_{00}(\omega, \mathbb{C})$ the minimal set of $c_{00}(\omega, \mathbb{C})$ satisfying the following properties:

- (1) For every $\alpha < \omega$, $e_\alpha^* \in W[(m_j^{-1}, n_j)]$. If $\phi \in W[(m_j^{-1}, n_j)]$ and $\theta = \lambda + i\mu$ is a complex number with λ and μ rationals and $|\theta| \leq 1$, $\theta\phi \in W[(m_j^{-1}, n_j)]$;
- (2) For every $\phi \in W[(m_j^{-1}, n_j)]$ and $E \subseteq \omega$, $E\phi \in W[(m_j^{-1}, n_j)]$;
- (3) For every $j \in \mathbb{N}$ and $\phi_1 < \dots < \phi_{n_j}$ in $W[(m_j^{-1}, n_j)]$, $(1/m_j) \sum_{i=1}^{n_j} \phi_i \in W[(m_j^{-1}, n_j)]$;
- (4) $W[(m_j^{-1}, n_j)]$ is closed under convex rational combinations.

Theorem 20 (Basic Inequality for R.I.S). *Let $(x_n)_n$ be a (C, ϵ) - R.I.S of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $(b_k)_k \in c_{00}(\mathbb{C}, \mathbb{N})$. Suppose that for some $j_0 \in \mathbb{N}$ we have that for every $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ with weight $w(f) = m_{j_0}$ and for every interval E of ω_1 ,*

$$\left| f \left(\sum_{k \in E} b_k x_k \right) \right| \leq C \left(\max_{k \in E} |b_k| + \epsilon \sum_{k \in E} |b_k| \right).$$

Then for every $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ of type I, there exist $g_1, g_2 \in c_{00}(\mathbb{C}, \mathbb{N})$ such that

$$\left| f \left(\sum_{k \in E} b_k x_k \right) \right| \leq C(g_1 + g_2) \left(\sum_{k \in E} |b_k| e_k \right),$$

where $g_1 = h_1$ ou $g_1 = e_t^ + h_1$, $t \notin \text{supp } h_1$, $e h_1 \in W[(m_j^{-1}, 4n_j)]$ such that $h_1 \in \text{conv}_{\mathbb{Q}} \left\{ h \in W[(m_j^{-1}, 4n_j)] : w(f) = w(h) \right\}$ and m_j does not appear as a weight of a node in the tree analysis of h_1 , and $\|g_2\|_\infty \leq \epsilon$.*

Proof. See [2, Section 8.2] □

The following results are consequences of the basic inequality. The proof of this properties in our case is the same as in [2].

Proposition 21. *Let $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ or $f \in W[(m_j^{-1}, 4n_j)]$ of type I. Consider $j \in \mathbb{N}$ and $l \in \left[\frac{n_j}{m_j}, n_j \right]$. Then for every set $F \subseteq c_{00}(\omega_1, \mathbb{C})$ of cardinality l ,*

$$\left| f \left(\frac{1}{l} \sum_{\alpha \in F} e_\alpha \right) \right| \leq \begin{cases} \frac{2}{w(f)m_j}, & \text{if } w(f) < m_j, \\ \frac{1}{w(f)} & \text{if } w(f) \geq m_j. \end{cases}$$

If the tree analysis of f does not contain nodes of weight m_j , then

$$\left| f \left(\frac{1}{l} \sum_{\alpha \in F} e_\alpha \right) \right| \leq \frac{2}{m_j^3}$$

Proof. [2, Proposition 4.6] □

Proposition 22. Let $(x_k)_k$ be a (C, ϵ) - R.I.S of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ with $\epsilon \leq \frac{1}{n_j}$, $l \in \left[\frac{n_j}{m_j}, n_j\right]$ and let $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ of type I. Then,

$$\left| f \left(\frac{1}{l} \sum_{k=1}^l x_k \right) \right| \leq \begin{cases} \frac{3C}{w(f)m_j}, & \text{if } w(f) < m_j \\ \frac{C}{w(f)} + \frac{2C}{n_j}, & \text{if } w(f) \geq m_j. \end{cases}$$

Consequently, if $(x_k)_{k=1}^l$ is a normalized (C, ϵ) - R.I.S with $\epsilon \leq \frac{1}{n_{2j}}$, $l \in \left[\frac{n_{2j}}{m_{2j}}, n_{2j}\right]$, then

$$\frac{1}{m_{2j}} \leq \left\| \frac{1}{l} \sum_{k=1}^l x_k \right\| \leq \frac{2C}{m_{2j}}.$$

Proof. Let $(x_k)_k$ be a (C, ϵ) - R.I.S and take $b = (\frac{1}{l}, \dots, \frac{1}{l}, 0, 0, \dots) \in c_{00}(\mathbb{N}, \mathbb{C})$. It follows from the basic inequality that for every $f \in \mathcal{K}_{\omega_1}(\mathbb{C})$ of type I, there exist $h_1 \in W[(m_j^{-1}, 4n_j)]$ with $\omega(h_1) = \omega(f)$, $t \in \mathbb{N}$ and $g_2 \in c_{00}(\mathbb{N})$ with $\|g\|_\infty \leq \epsilon$ such that

$$\left| f \left(\frac{1}{l} \sum_{k=1}^l x_k \right) \right| \leq C(e_t^* + h_1 + g_2) \left(\frac{1}{l} \sum_{k=1}^l e_k \right).$$

moreover,

$$\left| g_2 \left(\frac{1}{l} \sum_{k=1}^l e_k \right) \right| \leq \|g_2\|_\infty \left\| \frac{1}{l} \sum_{k \in E} e_k \right\|_1 \leq \epsilon \leq \frac{1}{n_j}.$$

Now by the estimatives on the auxiliary space $T[(m_j^{-1}, 4n_j)_j]$ of the Proposition 21, we have

- If $\omega(f) < m_j$,

$$\begin{aligned} \left| f \left(\frac{1}{l} \sum_{k=1}^l x_k \right) \right| &\leq C \left(\frac{1}{l} + \frac{2}{\omega(f)m_j} + \frac{1}{n_j} \right) \\ &\leq C \left(\frac{m_j}{n_j} + \frac{2}{\omega(f)m_j} + \frac{1}{n_j} \right) \\ &\leq \frac{3C}{\omega(f)m_j} \end{aligned}$$

- If $\omega(f) \geq m_j$,

$$\begin{aligned} \left| f \left(\frac{1}{l} \sum_{k=1}^l x_k \right) \right| &\leq C \left(\frac{1}{l} + \frac{C}{\omega(f)} + \frac{1}{n_j} \right) \\ &\leq \frac{C}{\omega(f)} + \frac{2C}{n_j} \end{aligned}$$

And notice

- $\frac{3C}{\omega(f)m_{2j}} \leq \frac{2C}{m_{2j}}$, if $\omega(f) < m_{2j}$,
- $\frac{C}{\omega(f)} + \frac{2C}{n_{2j}} \leq \frac{C}{m_{2j}} + \frac{C}{m_{2j}} = \frac{2C}{m_{2j}}$, if $\omega(f) \geq m_{2j}$.

We conclude from the fact that $\mathcal{K}_{\omega_1}(\mathbb{C})$ is the norming set:

$$\|(1/l) \sum_{k=1}^l x_k\| \leq 2C/m_{2j}.$$

For the proof the second part of the theorem, let $(x_k)_{k=1}^l$ be a normalized (C, ϵ) -R.I.S with $\epsilon \leq \frac{1}{n_{2j}}$, $l \in \left[\frac{n_{2j}}{m_{2j}}, n_{2j}\right]$. For every $k \leq l$, we consider $x_k^* \in \mathcal{K}_{\omega_1}(\mathbb{C})$, such that $x_k^*(x_k) = 1$ and $\text{ran } x_k^* \subseteq \text{ran } x_k$, then $x^* = \frac{1}{m_{2j}} \sum_{k=1}^l x_k^* \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and $x^* \left(\frac{1}{l} \sum_{k=1}^l x_k \right) = \frac{1}{m_{2j}}$. Hence, $\frac{1}{m_{2j}} \leq \left\| \frac{1}{l} \sum_{k=1}^l x_k \right\|$. \square

5.4. Proof Step II. Now we introduce another type of sequences in order to construct the conditional frame in $\mathfrak{X}_{\omega_1}(\mathbb{C})$. In fact, this space has no unconditional basic sequence.

Definition 23. A pair (x, ϕ) with $x \in \mathfrak{X}_{\omega_1}(\mathbb{C})$ and $\phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$, is called a (C, j) -exact pair when:

- (a) $\|x\| \leq C$, $\omega(\phi) = m_j$ and $\phi(x) = 1$.
- (b) For each $\psi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ of type I and $\omega(x) = m_i$, $i \neq j$, we have

$$|\psi(x)| \leq \begin{cases} \frac{2C}{m_i}, & \text{if } i < j \\ \frac{C}{m_j^2} & \text{if } i > j \end{cases}$$

Proposition 24. Let $(x_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then for every $j \in \mathbb{N}$, there exist (x, ϕ) such that $x \in \text{span}_{\mathbb{R}}(x_n)$, $\phi \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and (x, ϕ) is a $(6, 2j)$ -exact pair.

Proof. Fix $(x_n)_n$ a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and a positive integer j . By the Proposition 17 there exists $(y_n)_n$ a normalized $(3, 1/n_{2j})$ -R.I.S in $\text{span}_{\mathbb{R}}(x_n)$. For every $1 \leq i \leq n_{2j}$ and $\epsilon > 0$, we take $\phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $\phi_i(y_i) > 1 - \epsilon$, and $\phi_i < \phi_{i+1}$. Let $x = (m_{2j}/n_{2j}) \sum_{i=1}^{n_{2j}} y_i$ and $\phi = (1/m_{2j}) \sum_{i=1}^{n_{2j}} \phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C})$. By perturbing x by a rational coefficient on the support of some y_i we may assume that then $\phi(x) = 1$ and using Proposition 22 we conclude that (x, ϕ) is a $(6, 2j)$ -exact pair. \square

Definition 25. Let $j \in \mathbb{N}$. A sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ is called a $(1, j)$ -dependent sequence when

$$(DS. 1) \text{ supp } x_1 \cup \text{supp } \phi_1 < \dots < \text{supp } x_{n_{2j+1}} \cup \text{supp } \phi_{n_{2j+1}}$$

(DS. 2) The sequence $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$ is a $2j+1$ -special sequence.

(DS. 3) (x_i, ϕ_i) is a $(6, 2j_i)$ -exact pair. $\# \text{supp } x_i \leq m_{2j+1}/n_{2j+1}^2$ for every $1 \leq i \leq n_{2j+1}$

(DS. 4). For every $(2j+1)$ -special sequence $\Psi = (\psi_1, \dots, \psi_{n_{2j+1}})$ we have that

$$\bigcup_{k_{\Phi, \Psi} < i < \lambda_{\Phi, \Psi}} \text{supp } x_i \cap \bigcup_{k_{\Phi, \Psi} < i < \lambda_{\Phi, \Psi}} \text{supp } \psi_i = \emptyset.$$

where $k_{\Phi, \Psi}, \lambda_{\Phi, \Psi}$ are numbers introduced in Definition 13.

Proposition 26. For every normalized block sequence $(y_n)_n$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every natural number j there exists a $(1, j)$ -dependent sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ such that x_i is in the \mathbb{R} -span of $(y_n)_n$ for every $i = 1, \dots, n_{2j+1}$.

Proof. Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$ and $j \in \mathbb{N}$. We construct the sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ inductively. First using Proposition 24 we choose a $(6, 2j_1)$ -exact pair (x_1, ϕ_1) such that j_1 is even, $m_{2j_1} > n_{2j+1}^2$ and $x_1 \in \text{span}_{\mathbb{R}}(y_n)_n$. Assume that we have constructed $(x_1, \phi_1, \dots, x_{l-1}, \phi_{l-1})$ such that there exists (p_1, \dots, p_{l-1}) satisfying

- (1) $\text{supp } x_1 \cup \text{supp } \phi_1 < \dots < \text{supp } x_{l-1} \cup \text{supp } \phi_{l-1}$, where $x_i \in \text{span}_{\mathbb{R}}(y_n)_n$ and (x_i, ϕ_i) is a $(6, 2j_i)$ -exact pair.
- (2) For $1 < i \leq l-1$, $w(\phi_i) = \sigma_{\varrho}(\phi_1, w(\phi_1), p_1, \dots, \phi_{i-1}, w(\phi_{i-1}), p_{i-1})$.
- (3) For $1 \leq i < l-1$, $p_i \geq \max\{p_{i-1}, p_{F_i}\}$, where $F_i = \bigcup_{k=1}^i \text{supp } \phi_k \cup \text{supp } x_k$.

To complete the inductive construction choose $p_{l-1} \geq \max\{p_{l-2}, p_{F_{l-1}}, n_{2j+1}^2 \# \text{supp } x_{l-1}\}$ and $2j_l = \sigma_{\varrho}(\phi_1, w(\phi_1), p_1, \dots, \phi_{l-1}, w(\phi_{l-1}), p_{l-1})$. Hence take a $(6, 2j_l)$ -exact pair (x_l, ϕ_l) such that $x_l \in \text{span}_{\mathbb{R}}(y_n)_n$ and $\text{supp } x_{l-1} \cup \text{supp } \phi_{l-1} < \text{supp } x_l \cup \text{supp } \phi_l$. Notice that properties (DS.1), (DS.2) and (DS.3) are clear by definition of the sequence and (DS.4) follows from (3) and (TP.3). \square

Modifying a little the previous argument we obtain the following:

Proposition 27. For every two normalized block sequences $(y_n)_n$ and $(z_n)_n$ of $\mathfrak{X}_{\omega_1}(\mathbb{C})$, and every $j \in \mathbb{N}$ there exists a $(1, j)$ -dependent sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ such that $x_{2l-1} \in \text{span}_{\mathbb{R}}(y_n)$ and $x_{2l} \in \text{span}_{\mathbb{R}}(z_n)$ for every $l = 1, \dots, n_{2j+1}$. \square

Another consequence of the basic inequality is the following proposition.

Proposition 28. Let $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ be a $(1, j)$ -dependent sequence. Then:

- (1) $\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} x_i \right\| \geq \frac{1}{m_{2j+1}}$
- (2) $\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} x_i \right\| \leq \frac{1}{m_{2j+1}^3}$

Proof. The first inequality is clear since the functional $\phi = 1/m_{2j+1} \sum_{i=1}^{n_{2j+1}} \phi_i \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and $\phi(\sum_{i=1}^{n_{2j+1}} x_i) = n_{2j+1}/m_{2j+1}$. The second is obtained by the Basic Inequality. For the complete proof see [2, Proposition 3.7]. \square

We now can give a proof of Step II.

Proposition 29. *Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Then the closure of the real span of $(y_n)_n$ is H.I.*

Proof. Let $(y_n)_n$ be a normalized block sequence of $\mathfrak{X}_{\omega_1}(\mathbb{C})$. Fix $\epsilon > 0$ and two block subsequences $(z_n)_n$ and $(w_n)_n$ in $\text{span}_{\mathbb{R}}(y_n)_n$. Take an integer j such that $m_{2j+1}\epsilon > 1$. By Proposition 27 there exist a $(1, j)$ -dependent sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ such that $x_{2i-1} \in \text{span}_{\mathbb{R}}(z_n)$ and $x_{2i} \in \text{span}_{\mathbb{R}}(w_n)$. We define $z = (1/n_{2j+1}) \sum_{i=1}^{n_{2j+1}} x_i$ and $w = (1/n_{2j+1}) \sum_{i=1}^{n_{2j+1}} x_i$. Notice that $z \in \text{span}_{\mathbb{R}}(z_n)$ and $w \in \text{span}_{\mathbb{R}}(w_n)$. Then by Proposition 28 we get $\|z + w\| \geq 1/m_{2j+1}$ and $\|z - w\| \geq 1/m_{2j+1}^2$. Hence $\|z - w\| \leq \epsilon\|z + w\|$. \square

5.5. Proof of Step III.

Definition 30. *A sequence $(z_1, \phi_1, \dots, z_{n_{2j+1}}, \phi_{n_{2j+1}})$ is called a $(0, j)$ -dependent sequence when it satisfies the following conditions:*

- (0DS.1) *The sequence $\Phi = (\phi_1, \dots, \phi_{n_{2j+1}})$ is a $2j+1$ -special sequence and $\phi_i(z_k) = 0$ for every $1 \leq i, k \leq n_{2j+1}$.*
- (0DS.2) *There exists $\{\psi_1, \dots, \psi_{n_{2j+1}}\} \subseteq \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $w(\psi_i) = w(\phi_i)$, $\#\text{supp } z_i \leq w(\phi_{i+1})/n_{2j+1}^2$ and (z_i, ψ_i) is a $(6, 2j_1)$ -exact pair for every $1 \leq i \leq n_{2j+1}$.*
- (0DS.3) *If $H = (h_1, \dots, h_{n_{2j+1}})$ is an arbitrary $2j+1$ -special sequence, then*

$$\left(\bigcup_{\kappa_{\Phi, H} < i < \lambda_{\Phi, H}} \text{supp } z_i \right) \cap \left(\bigcup_{\kappa_{\Phi, H} < i < \lambda_{\Phi, H}} \text{supp } h_i \right) = \emptyset.$$

Proposition 31. *For every $(0, j)$ -dependent sequence $(x_1, \phi_1, \dots, x_{n_{2j+1}}, \phi_{n_{2j+1}})$ we have that*

$$\left\| \frac{1}{n_{2j+1}} \sum_{k=1}^{n_{2j+1}} x_k \right\| \leq \frac{1}{m_{2j+1}^2}.$$

Proof. [2, Proposition 5.23]. \square

Proposition 32. *Let $(y_n)_n$ be a (C, ϵ) -R.I.S, $Y = \overline{\text{span}}_{\mathbb{C}}(y_n)$, and $T : Y \rightarrow \mathfrak{X}_{\omega_1}(\mathbb{C})$ a \mathbb{R} -linear bounded operator. Then $\lim_{n \rightarrow \infty} d(Ty_n, \mathbb{C}y_n) = 0$.*

Proof. Suppose that $\lim_{n \rightarrow \infty} d(Ty_n, \mathbb{C}y_n) \neq 0$. Then there exists an infinite subset $B \subseteq \mathbb{N}$ such that $\inf_{n \in B} d(Ty_n, \mathbb{C}y_n) > 0$. We shall show that for every $\epsilon > 0$ there exists $y \in Y$ such that $\|y\| < \epsilon\|Ty\|$ and this is a contradiction.

Claim 1 There exists a limit ordinal γ_0 , $A \subseteq \mathbb{N}$ infinite and $\delta > 0$ such that

$$\inf_{n \in A} d(P_{\gamma_0} Ty_n, \mathbb{C}y_n) > \delta$$

To prove this claim we observe that

$$\gamma_0 = \min\{\gamma < \omega_1 : \exists A \in [\mathbb{N}]^\infty \inf_{n \in A} d(P_\gamma T y_n, \mathbb{C} y_n) > 0\}$$

is a limit ordinal. In fact, by the assumption the set on the right side is not empty. And if γ_0 is not limit, then we have $\gamma_0 = \beta + 1$. The sequence $(y_n)_n$ is weakly null (because $(e_\alpha)_\alpha$ is shrinking) and then

$$\lim_{n \rightarrow \infty} e_{\beta+1}^* T y_n = 0$$

And for large n and every $\lambda \in \mathbb{C}$

$$\begin{aligned} \|P_\beta T y_n - \lambda y_n\| &\geq \|P_{\beta+1} T y_n - \lambda y_n\| - \|e_{\beta+1}^* T y_n\| \\ &\geq \delta - |e_{\beta+1}^* T y_n| \geq \delta/2, \end{aligned}$$

which is a contradiction.

Claim 2 Fix γ_0 and $A \subseteq \mathbb{N}$ as in Claim 1. Then there exist a sequence $n_2 < n_3 < \dots$ in A , a sequence of functionals f_2, f_3, \dots in $\mathcal{K}_{\omega_1}(\mathbb{C})$ and a sequence of ordinals $\gamma_1 < \gamma_2 < \dots < \gamma_0$ such that

- (1) $d(P_{[\gamma_k, \gamma_{k+1}]} T y_{n_k}, \mathbb{C} y_{n_k}) \geq \delta/2$
- (2) $f_k T y_{n_k} \geq \delta/2$
- (3) $f_k(y_{n_k}) = 0$
- (4) $\text{ran } f_k \subseteq \text{ran } T y_{n_k}$
- (5) $\text{supp } f_k \cap \text{supp } y_{n_m} = \emptyset$ when $m \neq k$

To prove this claim, let $\xi = \sup \max \text{supp } y_n$. We analyze the three possibilities for ξ :

Case a.) $\xi < \gamma_0$. Let $n_1 = \min A$ and choose $\xi < \gamma_1 < \gamma_0$ such that

$$\|P_{\gamma_0} T y_{n_1} - P_{\gamma_1} T y_{n_1}\| < \delta/2,$$

hence, $d(P_{\gamma_1} T y_{n_1}, \mathbb{C} y_{n_1}) > \delta/2$. By minimality of γ_0 we have

$$\inf_{n \in A} d(P_{\gamma_1} T y_n, \mathbb{C} y_n) = 0,$$

then we can choose $n_2 > n_1$ in A such that $d(P_{\gamma_1} T y_{n_2}, \mathbb{C} y_{n_2}) < \delta/2$ and this implies that

$$d((P_{\gamma_0} - P_{\gamma_1}) T y_{n_2}, \mathbb{C} y_{n_2}) > \delta/2.$$

Approximating the vector $(P_{\gamma_0} - P_{\gamma_1}) T y_{n_2}$ choose $\gamma_0 > \gamma_2 > \gamma_1$ such that $\|(P_{\gamma_0} - P_{\gamma_2}) T y_{n_2}\|$ is so small in order to guarantee that

$$d(P_{[\gamma_1, \gamma_2]} T y_{n_2}, \mathbb{C} y_{n_2}) > \delta/2.$$

Using the complex Hahn-Banach theorem, there exists $g_2 \in B_{\mathfrak{X}_{\omega_1}^*}(\mathbb{C})$ such that

- $g_2(P_{[\gamma_1, \gamma_2]} T y_{n_2}) > \delta/2$
- $g_2(y_{n_2}) = 0$

and by Proposition 1 we can choose $h_2 \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $h_2(P_{[\gamma_1, \gamma_2]}Ty_{n_2}) > \delta/2$ and $h_2(y_{n_2})$ is arbitrarily small. Replacing h_2 by $\alpha h_2 + \beta k_2$ where $|\alpha| + |\beta| = 1$, $k_2(y_{n_2})$ is close enough to 1, and $k_2 \in \mathcal{K}_{\omega_1}(\mathbb{C})$ we may assume that $h_2(y_{n_2}) = 0$.

Let $f_2 = h_2 P_{[\gamma_1, \gamma_2] \cap \text{ran } Ty_{n_2}} \in \mathcal{K}_{\omega_1}(\mathbb{C})$. Again by minimality of γ_0 , there exists $n_3 > n_2$ in A such that $d(P_{\gamma_2}Ty_{n_3}, \mathbb{C}y_{n_3}) < \delta/2$ and we can choose $\gamma_0 > \gamma_3 > \gamma_2$ satisfying

$$d(P_{[\gamma_2, \gamma_3]}Ty_{n_3}, \mathbb{C}y_{n_3}) > \delta/2.$$

Again by Hahn-Banach and by Proposition 1 there exists a functional $h_3 \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that

- $h_3(P_{[\gamma_2, \gamma_3]}Ty_{n_3}) > \delta/2$
- $h_3(y_{n_3}) = 0$

then we define $f_3 = h_3 P_{[\gamma_2, \gamma_3] \cap \text{ran } Ty_{n_3}} \in \mathcal{K}_{\omega_1}(\mathbb{C})$. The previous argument gives us the way to construct the sequences of Claim 2. Properties (1)-(5) are easy to check, in particular property (5) is true because $\min \text{supp } f_k > \xi > \max \text{supp } y_{n_l}$ for every positive integers k, l .

Case b.) $\xi > \gamma_0$. In this case we start by picking $n_1 \in A$ such that $\min \text{supp } y_{n_1} > \gamma_0$. Then we repeat exactly the same argument that in case a.).

Case c.) $\xi = \gamma_0$. We basically repeat the same argument of the case a.) with the additional care of maintaining property (5) true. That is, each time we choose the ordinal γ_{k+1} (with $\gamma_0 > \gamma_{k+1} > \gamma_k$) we take it such that $\gamma_{k+1} > \max \text{supp } y_{n_{k+1}}$.

Claim 3 There exists a $(0, j)$ - dependent sequence $(z_1, \phi_1, \dots, z_{n_{2j+1}})$ such that

- $z_i \in X$ for every $1 \leq i \leq n_{2j+1}$
- $\text{ran } \phi_k \subseteq \text{ran } Ty_k$ and $\phi_k(Tz_k) > \delta/2$

Let j with $m_{2j+1} > 24/\epsilon\delta$. Choose j_1 even such that $m_{2j_1} > n_{2j_1}^2$ (see definition of special sequence) and $F_1 \subseteq A$ with $\#F_1 = n_{2j_1}$ such that $(y_{n_k})_{k \in F_1}$ is a $(3, 1/n_{2j_1}^2)$ -R.I.S. Then define

$$\phi_1 = \frac{1}{m_{2j_1}} \sum_{i \in F_1} f_i \in \mathcal{K}_{\omega_1}(\mathbb{C}) \text{ and } z_1 = \frac{m_{2j_1}}{n_{2j_1}} \sum_{k \in F_1} y_k$$

observe that $w(\phi_1) = m_{2j_1}$, $\phi_1(Tz_1) = \frac{1}{n_{2j_1}} \sum_{i \in F_1} f_i (\sum_{k \in F_1} Ty_k) > \delta/2$ and $\phi_1(z_1) = \frac{1}{n_{2j_1}} \sum_{i \in F_1} f_i (\sum_{k \in F_1} y_k) = 0$. Select

$$p_1 \geq \max\{p_\varrho(\text{supp } z_1 \cup \text{supp } Ty_1 \cup \text{supp } \phi_1), n_{2j_1}^2 \# \text{supp } z_1\}$$

denote $2j_2 = \sigma_\varrho(\phi_1, m_{2j_1}, p_1)$. Then take $F_2 \subseteq A$ with $\#F_2 = n_{2j_2}$ and $F_2 > F_1$ such that $(y_k)_{k \in F_2}$ is $(3, 1/n_{2j_2}^2)$ -R.I.S and define

$$\phi_2 = \frac{1}{m_{2j_2}} \sum_{i \in F_2} f_i \in \mathcal{K}_{\omega_1}(\mathbb{C}) \text{ and } z_2 = \frac{m_{2j_2}}{n_{2j_2}} \sum_{k \in F_2} y_k$$

So we have $\phi_1 < \phi_2$, $\phi_2(Tz_2) > \delta$ and $\phi_2(z_1) = \phi_2(z_2) = 0$. Pick

$$p_2 \geq \max\{p_1, p_\varrho(\text{supp } z_1 \cup \text{supp } z_2 \cup \text{supp } Tz_1 \cup \text{supp } Tz_2 \cup \text{supp } \phi_1 \cup \text{supp } \phi_2), n_{2j+1}^2 \# \text{supp } z_2\}$$

and set $2j_3 = \sigma_\varrho(\phi_1, m_{2j_1}, p_1, \phi_2, m_{2j_2}, p_2)$. Continuing with this procedure we form a sequence $(z_1, \phi_1, \dots, z_{n_{2j+1}}, \phi_{n_{2j+1}})$. Now we check that this is a $(0, j)$ -dependent sequence.

Property (0DS.1) is clear, because of the construction of the functionals their weights satisfies $w(\phi_{i+1}) = m_{\sigma_\varrho(\Phi_i)}$ where $\Phi_i = (\phi_1, w(\phi_1), p_1, \dots, \phi_i, w(\phi_i), p_i)$.

Property (0DS.2) We proceed to the construction of the sequence $\{\psi_1, \dots, \psi_{n_{2j+1}}\}$ in $\mathcal{K}_{\omega_1}(\mathbb{C})$ such that (z_i, ψ_i) is a $(6, 2j_i)$ -exact pair and $w(\psi_i) = w(\phi_i)$ for every $1 \leq i \leq n_{2j+1}$. The other condition $\# \text{supp } z_i \leq w(\phi_{i+1})/n_{2j+1}^2$ is already obtained by the construction of the weights. For each z_i there exists a subset $F_i \subseteq A$ with $\#F_i = n_{2j_i}$ such that $z_i = (m_{2j_i}/n_{2j_i}) \sum_{k \in F_i} y_{n_k}$ where $(y_{n_k})_{k \in F_i}$ is a $(3, 1/n_{2j_i}^2)$ R.I.S. Now we follow the same arguments as in Proposition 24. For every $k \in F_i$ we take $f_{n_k} \in \mathcal{K}_{\omega_1}(\mathbb{C})$ such that $f_{n_k}(y_{n_k}) = 1$ and $f_{n_k} < f_{n_{k+1}}$. Then $\psi_i = (1/m_{2j_i}) \sum_{k \in F_i} f_{n_k} \in \mathcal{K}_{\omega_1}(\mathbb{C})$ and (z_i, ϕ_i) is a $(6, 2j_i)$ -exact pair.

Property (0DS.3) Let $H = (h_1, \dots, h_{n_{2j+1}})$ be an arbitrary $2j+1$ -special sequence. We consider two cases: a) Suppose that $\max \text{supp } z_k \leq \max \text{supp } \phi_k$ for every $1 \leq k \leq n_{2j+1}$. Then $\text{supp } z_k \subseteq \text{supp } \overline{\phi_{\lambda_{\Phi, H^{-1}}}}^{p_{\lambda_{\Phi, H^{-1}}}}$ for every $\kappa_{\Phi, H} < k < \lambda_{\Phi, H}$. Then for the second part of (TP.3) we obtain the desired result. (b) Suppose that $\max \text{supp } \phi_k \leq \max \text{supp } z_k$ for every $1 \leq k \leq n_{2j+1}$. Then $\text{supp } \phi_k \subseteq \text{supp } \overline{z_{\lambda_{\Phi, H^{-1}}}}^{p_{\lambda_{\Phi, H^{-1}}}}$ for every $\kappa_{\Phi, H} < k < \lambda_{\Phi, H}$, and the result follows from the first part of (TP3).

Fix a $(0, j)$ -dependent sequence as obtained in the previous claim, and define

$$z = (1/n_{2j+1}) \sum_{k=1}^{n_{2j+1}} z_k \text{ and } \phi = (1/m_{2j+1}) \sum_{k=1}^{n_{2j+1}} \phi_k.$$

Then $\phi(Tz) = (1/n_{2j+1}) \sum_{k=1}^{n_{2j+1}} \phi_k(Tz) \geq \delta/m_{2j+1}$ and $\|z\| \leq 12/m_{2j+1}^2$. Hence, $\|Tz\| \geq \delta/m_{2j+1} \geq \delta m_{2j+1} \|z\|/12 > \epsilon \|z\|$ and this completes the proof. \square

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